An overview of Hladký et al's (2021) Work on Inhomogeneous W-Random Graphs

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Paper discussed

Hladký, Jan, et al. “A Limit Theorem for Small Cliques in Inhomogeneous Random Graphs.” Journal of Graph Theory, vol. 97, no. 4, 2021, pp. 578–599, https://arxiv.org/abs/1903.10570

Overview

- Graphons
- W-Random Graphs
- Graph Homomorphisms
- Homomorphism Density
- K_r-free & K_r-complete graphons (Theorem 1.2a)
- Conditional homomorphism density
- K_r-regular graphons
- Statement of Theorems 1.2b & 1.2c
- Proof Idea for Theorems
- Extensions & Concluding Remarks

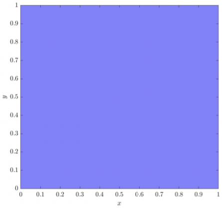
Graphons

Definition

A **graphon** is a bounded, symmetric and measurable function

$$W : [0,1]^2 \to [0,1] \quad \text{where } W(x,y) = W(y,x)\ \forall\ x,y \in [0,1]$$

Uniform (Erdős-Rényi)



$W(u,v) = p$

Figure: A constant graphon (Ribeiro 2021)

Graphons

- Graphons $\approx$ weighted symmetric graphs with uncountably many vertices
- Graphons $\approx$ limit of graph sequences
- If G is an unweighted graph, fix $w_e = 1$ for each edge e.

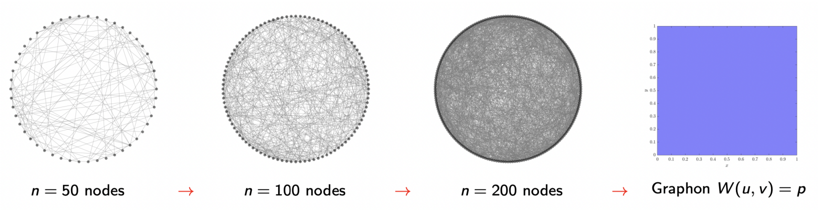


Figure: Sequence of random graphs sampled from a constant graphon (Ribeiro 2021)

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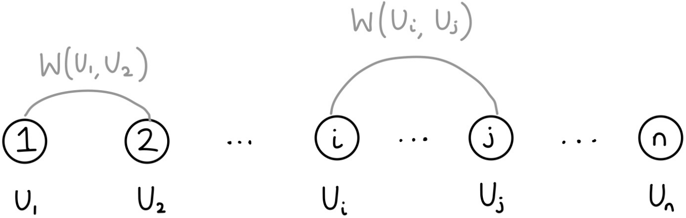

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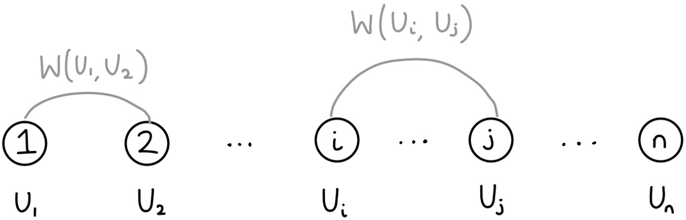

- $W(x, y) \equiv p \Rightarrow \mathbb{G}(n, W)$ equivalent to Erdős–Rényi $\mathbb{G}(n, p)$ random graph

Graph Homomorphisms

Definition

For graphs $F = (V', E')$ & $G = (V, E)$,
a **graph homomorphism** from F to G is a map

$$\beta : V' \to V \quad \text{s.t. if } (i, j) \in E', \text{ then } (\beta(i), \beta(j)) \in E$$

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- Adjacency-preserving map
- Homomorphism $K_r \to G \Rightarrow G$ contains an r-clique

Graph Homomorphisms

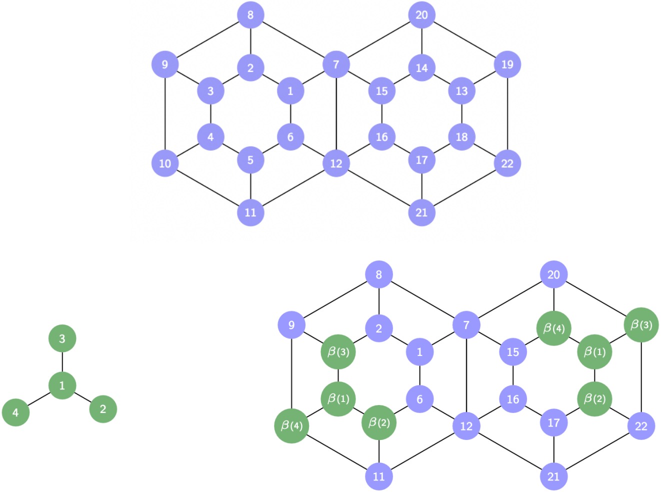

Figure: Example of multiple homomorphisms $F \to G$ (Ribeiro 2021)

Homomorphism Density for Weighted Graphs

Definition

For $G = (V, E)$ on n nodes & $F = (V', E')$ on k nodes, the **homomorphism density** of F in G is:

$$t(F, G) = \frac{1}{n^k} \sum_{\substack{\beta: V' \to V \\ \text{graph hom.}}} \left(\prod_{(i,j) \in E'} [A]_{\beta(i), \beta(j)} \right)$$

where A is the adjacency matrix of G.

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where A is the adjacency matrix of G.

- Weight each homomorphism $\beta : V' \to V$ by the product of edge weights in the image of β

Homomorphism Densities for Graphons

Definition

For a graphon W & multigraph $H = (V, E)$ on n nodes, the **homomorphism density** of H in W is:

$$t(H, W) = \int_{[0,1]^n} \prod_{(i,j) \in E} W(x_i, x_j) \prod_{i \in V} dx_i$$

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- For a clique K_r ($r \geq 2$), the homomorphism density can be defined as:

$$t_r := t(K_r, W) = \mathbb{E}\left[\prod_{(i,j)\in E} W(U_i, U_j)\right]$$

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- Akin to weighted graph definition, where $W(x_i, x_j)$ is the weight of edge (i, j)

K_r-free and K_r-complete graphons

Definition

A graphon W is K_r-**free** if $t(K_r, W) = 0$ &
K_r-**complete** if $t(K_r, W) = 1$ almost everywhere.

K_r-free and K_r-complete graphons

Definition

A graphon W is K_r-**free** if $t(K_r, W) = 0$ &
K_r-**complete** if $t(K_r, W) = 1$ almost everywhere.

- Let $X_{n,r}$ = no. of r-cliques in $\mathbb{G}(n, W)$.

Theorem

If W is K_r-free or K_r-complete,
then almost surely $X_{n,r} = 0$ or $X_{n,r} = \binom{n}{r}$ respectively.

Statement of Theorem 1.2a

- Consider the graphon

$$W(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

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$$W(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

- W is K_r-complete $\Rightarrow$ There are $\binom{n}{r}$ r-cliques

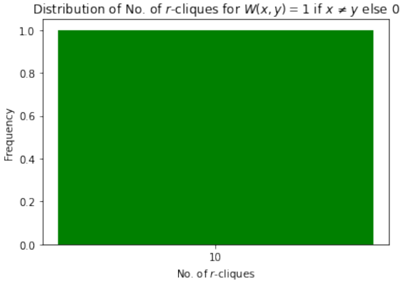

Figure: No. of 3-cliques in $\mathbb{G}(5, W)$ (sampled 1000 times)

Conditional Homomorphism Density

Definition

Let H be a graph with vertex set $[k]$ where nodes in $J \subseteq [k]$ are **marked**. For a vector of values $\mathbf{x} = (x_j)_{j \in J} \in [0,1]^{|J|}$, the **conditional density** is:

$$t_{\mathbf{x}}(H, W) = \mathbb{E}\left[\prod_{\{i,j\} \in E(H)} W(U_i, U_j) \;\middle|\; U_j = x_j : j \in J \right]$$

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- $t_{\mathbf{x}}(K_r, W)$ is the conditional probability that $\mathbb{G}(r, W) = K_r$ whenever each node $j \in J$ has type x_j

Degree Function of a Graphon

Definition

For a graphon W, the **degree function** $\deg_W : [0,1] \to [0,1]$ is:

$$\deg_W(x) = \int_0^1 W(x,y)\,dy$$

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- In an Erdős–Rényi random graph $\mathbb{G}(n,p)$, a node has expected degree $(n-1)\cdot p$
- In $\mathbb{G}(n,W)$, a node with type $x \in [0,1]$ has expected degree is $(n-1)\cdot \deg_W(x)$

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- In $\mathbb{G}(n,W)$, a node with type $x \in [0,1]$ has expected degree is $(n-1) \cdot \deg_W(x)$

Definition

Say that a graphon W is **regular** if $\deg_W(x) \equiv d$ for some constant $d \in [0,1]$.

K_r-regular graphons

- Let $K_r^\bullet := K_r$ with one marked node, with conditional density $t_x(K_r^\bullet, W)$

K_r-regular graphons

- Let $K_r^\bullet := K_r$ with one marked node, with conditional density $t_x(K_r^\bullet, W)$

Definition (Equation 8, Hladký et al. 2021)

A graphon W is K_r-**regular** if for almost every $x \in [0,1]$, we have:

$$t_x(K_r^\bullet, W) = t(K_r, W)$$

- K_r-regularity = generalization of graph regularity

Statement of Theorem 1.2b

- If W is not K_r-regular, then the no. of r-cliques exhibits fluctuations that are asymptotically Gaussian.

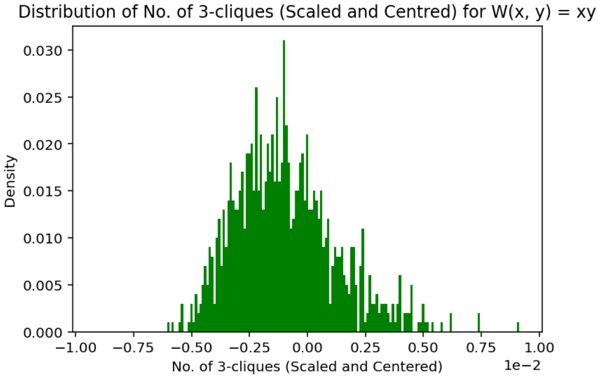
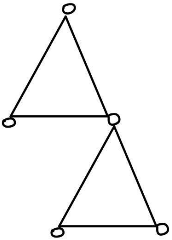


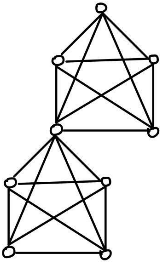
Figure: Numerical simulations (1000 iterations) for the distribution of $\frac{X_{n,r} - \mathbb{E}[X_{n,r}]}{n^{r-1/2}}$ in $\mathbb{G}(100, W)$, where $W(x, y) = xy$, $r = 3$

Statement of Theorem 1.2b

- Let $K_r \ominus_1 K_r$ = simple graph on $2r - 1$ nodes with two copies of K_r sharing one node



$K_3 \ominus_1 K_3$



$K_5 \ominus_1 K_5$

Statement of Theorem 1.2b

Theorem

If W is not K_r-regular, then:

$$\frac{X_{n,r} - \binom{n}{r} t_r}{n^{r-1/2}} \xrightarrow{d} \hat{\sigma}_{r,W} \cdot Z$$

where $Z \sim N(0,1)$ & $\hat{\sigma}_{r,W} = \frac{1}{(r-1)!}\left(t(K_r \ominus_1 K_r, W) - t_r^2\right)^{1/2} > 0$

Statement of Theorem 1.2b

Theorem

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- View $\hat{\sigma}_{r,W}$ as the *scaled* variance of $X_{n,r}$, where:
  - $t(K_r \ominus_1 K_r, W) \approx \mathbb{E}[X_{n,r}^2]$
  - $t_r^2 \approx \mathbb{E}[X_{n,r}]^2$

Motivating Example for Theorem 1.2c

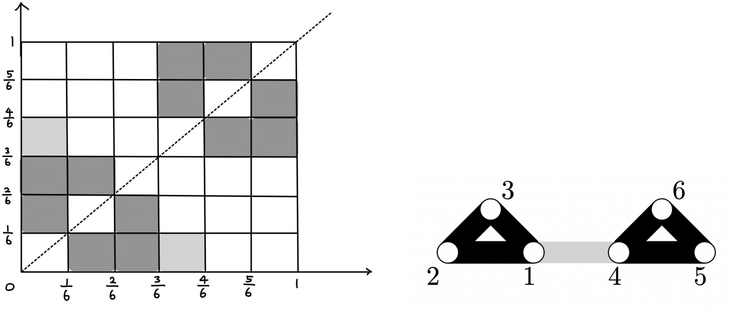

Figure: K_3-regular graphon where $\frac{X_{n,r} - \mathbb{E}[X_{n,r}]}{n^{r-1}}$ follows a chi-square distribution (right picture from Hladký et al.)

Motivating Example for Theorem 1.2c

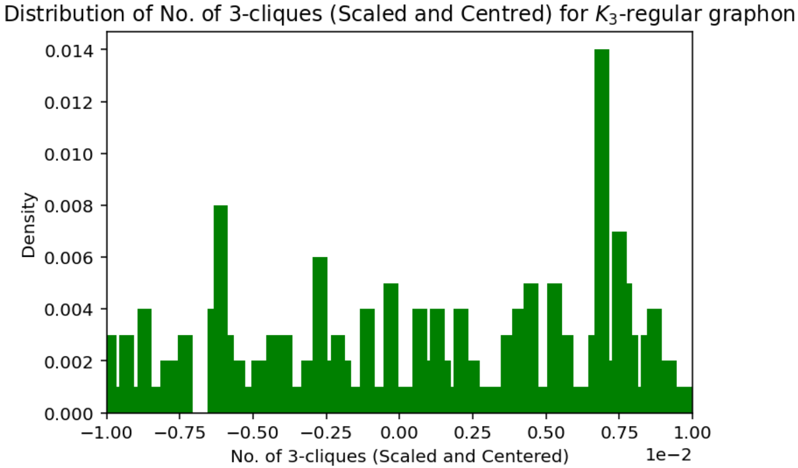

Figure: Simulated no. of 3-cliques (1000 iterations)

(Simplified) Statement of Theorem 1.2(c)

- Let W be a K_r-regular graphon

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- Let W be a K_r-regular graphon
- Then $t_x(K_r, W) = t(K_r, W)$ for almost all $x \in [0, 1]$

(Simplified) Statement of Theorem 1.2(c)

- Let W be a K_r-regular graphon
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Theorem

If $t(K_r, W)$ *is constant &* $t(K_r, W) \notin \{0,1\}$, *then* $\exists\ c_0, c_1, \ldots \in \mathbb{R}$ *s.t.* $\sum_i c_i^2 \in (0,\infty)$ *and:*

$$\frac{X_{n,r} - \mathbb{E}[X_{n,r}]}{n^{r-1}} \xrightarrow{\mathbf{d}} c_o Z_0 + \sum_{i\geq 1} c_i (Z_i^2 - 1)$$

where $Z_0, Z_1, \ldots$ *are independent standard normal.*

Theorem 1.2b Proof Idea

- Dependency Graphs
- Wasserstein Distance

Dependency Graphs

- Given a collection of random variables (RVs) $(Y_i)_{i \in I}$ for some index set I, create a **dependency graph** $\mathcal{G}$ with vertex set I

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Dependency Graphs

- Given a collection of random variables (RVs) $(Y_i)_{i \in I}$ for some index set I, create a **dependency graph** $\mathcal{G}$ with vertex set I
- For each vertex $i \in I$, let N_i denote the neighborhood of $i \in \mathcal{G}$
- Construct $\mathcal{G}$ such that:

$$\forall\ i \in I,\ \text{the random variable } Y_i \text{ is independent of } \{Y_j\}_{j \notin N_i}$$

Dependency Graph Example

$$
\begin{aligned}
Y_1, Z, Z', Y_4, Y_5 &\sim N(0,1) \quad \text{(i.i.d standard normal)} \\
Y_2 &:= \tfrac{1}{\sqrt{2}}(Y_1 + Z) \\
Y_3 &:= \tfrac{1}{\sqrt{2}}(Y_1 + Z') \\
Y_2, Y_3 &\sim N(0,1)
\end{aligned}
$$

Dependency Graph Example

$$Y_1, Z, Z', Y_4, Y_5 \sim N(0,1) \quad \text{(i.i.d standard normal)}$$

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- Y_1, Y_2, Y_3 are dependent standard normal
- $\{Y_1, Y_2, Y_3\}$ are independent of Y_4 and Y_5, where $Y_4 \perp\!\!\!\perp Y_5$.

Dependency Graph Example

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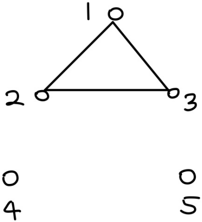
- Y_1, Y_2, Y_3 are dependent standard normal
- $\{Y_1, Y_2, Y_3\}$ are independent of Y_4 and Y_5, where $Y_4 \perp\!\!\!\perp Y_5$.
- $\mathcal{G}$ is given by:



Wasserstein Distance

- Let $d_{Wass}(X, Y)$ be the Wasserstein distance between two RVs X, Y

Wasserstein Distance

- Let $d_{Wass}(X, Y)$ be the Wasserstein distance between two RVs X, Y
- For $Z \sim N(0, 1)$ and a sequence of RVs $\{X_n\}_{n=1}^{\infty}$, consider the well-known convergence result:

$$d_{Wass}(X_n, Z) \to 0 \implies X_n \xrightarrow{d} Z$$

Theorem 1.2b Proof Idea

- **Setup**: Create a dependency graph $\mathcal{G}$ for the collection of RVs $(Y_R)_R$, where $R \subset [n], \ |R| = r < n$

Theorem 1.2b Proof Idea

- **Setup**: Create a dependency graph $\mathcal{G}$ for the collection of RVs $(Y_R)_R$, where $R \subset [n], \ |R| = r < n$
- In $\mathcal{G}$, edges $(R_1, R_2) \longleftrightarrow$ non-disjoint subsets $R_1, R_2 \subset [n]$
- How should we define the variables Y_R?

Theorem 1.2b Proof Idea

- Let $I_R := \mathbf{1}(R \text{ induces a clique in } \mathbb{G}(n, W))$

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- Let $I_R := \mathbf{1}(R \text{ induces a clique in } \mathbb{G}(n, W))$

- Let $Y_R := I_R - \mathbb{E}[I_R] = I_R - t_r$
- Then $\sum_R Y_R = X_{n,r} - \binom{n}{r} t_r$

Theorem 1.2b Proof Idea

- Let $I_R := \mathbf{1}(R \text{ induces a clique in } \mathbb{G}(n, W))$
- Let $Y_R := I_R - \mathbb{E}[I_R] = I_R - t_r$
- Then $\sum_R Y_R = X_{n,r} - \binom{n}{r} t_r$
- The proof consists of two steps:
  1. Bound the maximum degree of any node in $\mathcal{G}$
  2. Compute the asymptotics of the variance of $\sum_R Y_R$

Theorem 1.2b Proof Idea

- **Step 1:** In $\mathcal{G}$, each neighbourhood N_R has the same size

$$\sum_{l=1}^{r} \binom{r}{l} \binom{n-r}{r-l} = O(n^{r-1})$$

Theorem 1.2b Proof Idea

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Theorem 1.2b Proof Idea

- Let $\sigma_n^2 = \text{Var}\left[\sum_R Y_R\right]$
- **Step 2:** Show that $d_{Wass}\left(\frac{\sum_R Y_R}{\sigma_n}, Z\right) = O(n^{-1/2})$
- Hence $\frac{\sum_R Y_R}{\sigma_n} \xrightarrow{d} Z$

Theorem 1.2b Proof Idea

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- Hence $\frac{\sum_R Y_R}{\sigma_n} \xrightarrow{d} Z$
- Since $\sum_R Y_R = X_{n,r} - \binom{n}{r} t_r$, this completes the proof.

Theorem 1.2c Proof Idea

- **Objective**: Use the method of moments to establish distributional convergence of $\frac{X_{n,r}-\binom{n}{r}t_r}{n^{r-1}}$
- Computing coefficients involves examining the isomorphism classes of hypergraphs induced by collections of vertex subsets of $\mathbb{G}(n, W)$

Extensions

- Bhattacharya, Chatterjee & Janson (2022) extended these results for general subgraphs H in W-random graphs
- Analogous notion of H-**regular graphons**:
  - If W is *not* H-regular, then the distribution of $X_n(H, W)$ is asymptotically Gaussian
  - If W is H-regular, then the limiting distribution of $X_n(H, W)$ consists of a Gaussian term and a Chi-squared term

Extensions

- Kaur & Röllin (2021) provide a central limit theorem for *centred* subgraph counts in W-random graphs, demonstrating distributional convergence to Gaussians
- Developed test statistics for determining the presence of certain subgraphs (eg. two edges sharing a common vertex)

Concluding Remarks

- We study a limit theorem for complete subgraph counts in W-random graphs, which exhibits normal or chi-square behavior

Concluding Remarks

- We study a limit theorem for complete subgraph counts in W-random graphs, which exhibits normal or chi-square behavior
- Open problems:
  - Find the no. of cliques & other subgraphs in *sparse* $\mathbb{G}(n, p \cdot W)$ where $p = p(n) \to 0$ as $n \to \infty$
  - Prove analogous results for W-random hypergraphs

Acknowledgements

Thank you for listening!

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Appendices

In the following slides, we discuss the high-level idea for the proofs for Theorems 1.2b-c in greater detail.

r-Uniform Hypergraphs, Clique Graphs

Definition

For $r \geq 2$, a r-**uniform hypergraph** $\mathcal{H}$ on a vertex set V is a collection of r-element subsets (**hyperedges**) of V.

Definition

Given a hypergraph $\mathcal{H}$, the **graph associated with** $\mathcal{H}$ (**clique graph** of $\mathcal{H}$) is a graph on the same vertex set, where each hyperedge S of $\mathcal{H}$ is replaced by a clique on S, with multiple edges replaced by single edges.

Loose Cycles (Hypergraph version of cycles)

- For $l \geq 2$, let $C_l^{(r)}$ be a r-uniform hypergraph with l hyperedges.
- To construct $C_l^{(r)}$, take the cycle graph C_l, and for each edge, insert an additional $r-2$ nodes, where all $l(r-2)$ new nodes are distinct.
- Then let $G_{l,r}$ be the graph associated with $C_l^{(r)}$.

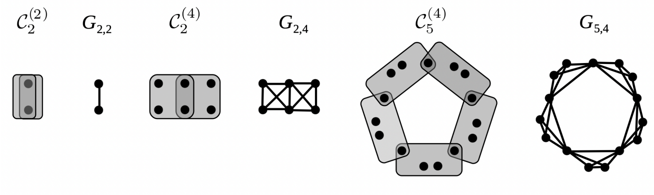


Figure: Examples of hypergraphs $C_l^{(r)}$ and their associated graphs $G_{l,r}$ (Hladký et al. 2021)

Spectrum of a Graphon

- Each graphon has an associated integral linear operator $T_W : L^2[0,1] \to L^2[0,1]$, where $(T_W f)(x) = \int_0^1 W(x,y) f(y)\, dy$
- T_W is a Hilbert-Schmidt operator and that there are countably many non-zero real eigenvalues associated with W.
- It can be shown that if W is a regular graphon, i.e. $\deg_W(x) \equiv d$ for some constant d, then W has an eigenfunction $f \equiv 1$ with associated eigenvalue d.
- Let $^{-}(W)$ be the multiset of eigenvalues of W, where the multiplicity of the eigenvalue d is decreased by 1.

The Graphon $V_W^{(r)}$

- For any graphon W and $r \geq 2$, define the graphon $V_W^{(r)}$ as:

$$V_W^{(r)}(x, y) = t_{x,y}(K_r^{\bullet\bullet}, W)$$

- View $V_W^{(r)}(x, y)$ as the conditional density of r-cliques containing nodes with types x, y
- It can be shown that if W is K_r-regular $\iff$ $V_W^{(r)}$ is regular

Full Statement of Theorem 1.2(c)

- Suppose W is a K_r-regular graphon that is neither K_r-free nor complete.
- Recall that $X_{n,r}$ denotes the no. of r-cliques in $\mathbb{G}(n, W)$. Then:

Theorem (Theorem 1.2c (abridged), Hladký et al. 2021)

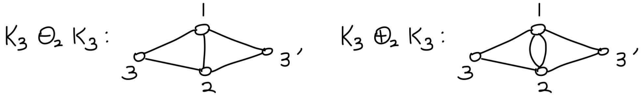
Let $r \geq 2$ and set $t_r = t(K_r, W)$. Then:

$$\frac{X_{n,r} - \binom{n}{r} t_r}{n^{r-1}} \xrightarrow{d} \sigma_{r,W} \cdot Z + \frac{1}{2(r-2)!} \sum_{\lambda \in Spec^-(V_W^{(r)})} \lambda (Z_\lambda^2 - 1)$$

where Z and $(Z_\lambda)_{\lambda \in Spec^-(V_W^{(r)})}$ are independent standard normal.

The Parameter $\sigma^2_{r,W}$

- Let $K_r \oplus_2 K_r$ denote the simple graph consisting of two r-cliques sharing 2 nodes (total of $2r - 2$ nodes)
- Let $K_r \ominus_2 K_r$ denote the multigraph obtained from $K_r \oplus_2 K_r$ where we duplicate the shared edge.



- (Equation 9, Hladký et al. 2021) We have that:

$$\begin{aligned} t_{x,y}(K_r \oplus_2 K_r, W) &= W(x,y)\, t_{x,y}(K_r \ominus_2 K_r, W) \\ &= (t_{x,y}(K_r^{\bullet\bullet}, W))^2 \\ &= (V_W^{(r)}(x,y))^2 \end{aligned}$$

Then define:

$$\sigma^2_{r,W} := \frac{1}{2((r-2)!)^2}\left(t(K_r \ominus_2 K_r, W) - t(K_r \oplus_2 K_r, W)\right)$$

Theorem 1.2c Proof Idea

- Analyse the structure of tuples $(R_1, \ldots, R_m)$ where each R_i is a subset of vertices of $\mathbb{G}(n, W)$
- Let $\mathfrak{X}(n, r, m)$ be the set of m-tuples where $\exists i \in [m]$ such that $|R_i \cap (\cup_{j \neq i} R_j)| \leq 1$
- Let $\Delta(R_1, \ldots, R_m) := \mathbb{E}\big[\prod_{i=1}^{m} (I_{R_i} - t_r)\big]$, and show that $\Delta(R_1, \ldots, R_m) = 0$ for all tuples in $\mathfrak{X}(n, r, m)$.
- Let $\mathfrak{F}(n, r, m)$ be tuples *not* in $\mathfrak{X}(n, r, m)$, where the corresponding hypergraph $\mathcal{H}$ has $(r-1)m$ nodes. One can show that such an $\mathcal{H}$ is a union of vertex-disjoint loose cycles.

Theorem 1.2c Proof Idea (cont.)

- Isomorphism classes of $\mathcal{H}$ can be encoded by a vector $\mathbf{k}$ where i-th component = no. of loose cycles of length i, where $\mathcal{H}_{\mathbf{k}}^{(r)}$ is the hypergraph formed by k_i copies of the loose cycle $C_i^{(r)}$
- Claim 4.3: Show that the contribution $\Delta(R_1, \ldots, R_m)$ for each tuple is the same, and obtain an explicit expression for the contribution.
- Claim 4.4: Count the no. of tuples in the isomorphism classes of $\mathcal{H}$.

Theorem 1.2c Proof Idea (cont.)

- Claim 4.5: Express $\mathbb{E}[(X_{n,r} - \binom{n}{r} t_r)^m]$ as a formal power series $f(x)$, and use results from previous claims to compute coefficients.
- Claim 4.6: Show that the MGF of Y is equal to $f(x)$ within a neighborhood of zero. (This verifies that the MGF of Y is finite in this neighborhood, so the distribution of Y is determined by its moments.)